Calculating the Statistics of Forced Response of a Mistuned Bladed Disk Assembly

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An analytical technique has been developed to calculate the statistics of the forced response of a mistuned bladed disk assembly. The modal mass and stiffness of each blade have been assumed to be random variables with Gaussian distribution. The comparison with results from numerical simulations indicate that the statistics of distributions of the blades' amplitudes predicted by the analytical technique are acceptable.

Nomenclature

A_j = amplitude of the jth mass (mistuned system)
a_{ij} = elements of matrix H^{-1}
B_j = amplitude of the jth mass (tuned system)
C = damping matrix
$E(\ )$ = expected value
f_j = force on the jth mass
f_0 = magnitude of the sinusoidal excitation
$g(\ )$ = probability density function
H = matrix with elements of M_t, K_t, and C
I = identity matrix
K = stiffness matrix (mistuned system)
K_t = stiffness matrix (tuned system)
k_e = equivalent stiffness
k_t = modal stiffness of each blade (tuned system)
M = mass matrix (mistuned system)
M_t = mass matrix (tuned system)
m_t = modal mass of each blade
N = number of degrees of freedom
NB = number of blades
NS = number of simulations
Q = matrix with elements of δK
R = matrix with elements of $\omega^2 \delta M$
U = vector defined by Eq. (10)
u_j = jth element of U
V = vector defined by Eq. (11)
v_j = jth element of V
X = response vector (mistuned system)
X_t = response vector (tuned system)
x_j = jth element of X
γ_j = correlation coefficient between u_{2j} and u_{2j-1}
ϵ_j = phase of δx_j
δA_j = amplitude of δx_j
δX = $X - X_t$
δx_j = jth element of δx
σ_j = standard deviation of u_j
ϕ_j = phase of x_j
ω = excitation frequency

I. Introduction

IN this paper, an analytical technique has been developed to calculate the statistics of the forced response of a mistuned bladed disk assembly. Mistuning refers to variations in the modal properties of the blades caused by minor differences in their geometries that arise during the manufacturing process. It has received wide attention in the existing literature[1-8] because even a small amount of mistuning can lead to a large variation in the blades' amplitudes within the same assembly. In fact, the amplitude of vibration of the worst blade may often be as high as one and one-half times the amplitude of a blade of a perfectly tuned system. Consequently, the fatigue life of a blade on a mistuned disk may be significantly lower than that predicted on the basis of a perfectly tuned system. Furthermore, the deviation of modal properties of blades from their nominal values are statistically distributed. As a result, there exists a large number of possible sets of blades' modal parameters that a bladed disk assembly can have. Since the maximum amplitude will be different for each such set of the blades' modal parameters, it is important for the designer to know the statistics of forced response, particularly the probability that the maximum amplitude on any disk would exceed a critical value.

Except for the paper by Huang,[9] the statistics in the published literature of forced response of a mistuned bladed disk assembly have been generated by numerical simulations.[6,7] The modal parameters of each blade of the assembly are chosen randomly from a large population with specified mean and standard deviation. Since the steady-state response of the resulting system is easily obtained, the probability distribution of amplitudes of vibrations can be generated by considering a large number of rotor stages with different sets of the blades' modal parameters. In principle, the numerical simulation of the responses of various mistuned bladed disk assemblies is straightforward. However, this approach is cumbersome and expensive in terms of computer time. To gain one additional significant figure in a result requires a hundredfold increase in cost.[10] The analytical approach used by Huang[9] can yield the mean and variance of the statistical distribution of the blades' amplitudes. However, this analysis is valid only for rotor stages with closely spaced blades. In addition, there is no attempt to either characterize the probability density functions of the amplitudes or calculate the probability that the maximum amplitude on any disk would be below a critical value.

The analytical technique presented here yields not only complete information about the probability density functions of the blades' amplitudes, but also has the advantage of being applicable to a system with any number of blades. Using this technique, the probability that a blade's amplitude is less than a certain value can be easily calculated. However, the accuracy of the method depends on the amount of damping in the system. The distribution of modal parameters of each blade has been assumed to be Gaussian in this paper. The extension of this technique to a non-Gaussian distribution of the blades' modal parameters is currently under development. First, the development of analytical method is pre-

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sented in detail. Next, the probability density functions obtained from this technique are compared with those from numerical simulations for a mistuned eight bladed disk. The statistics of the blades' amplitudes predicted by the analytical technique are found to be acceptable for a small amount of mistuning at damping levels that are typical of those found in applications.

II. Development of the Analytical Technique

Let M, K, and C be the mass, stiffness, and damping matrices, respectively. If the mass and stiffness matrices for the corresponding tuned system are M_t and K_t,

$$M = M_t + \delta M$$

$$K = K_t + \delta K \tag{1}$$

where δM and δK are deviations in the mass and stiffness matrices due to mistuning. It will be assumed that the probability density function of each of the nonzero elements of δM and δK is Gaussian with zero mean and are independent.

The external force on each blade is assumed to represent a particular engine order excitation and, as a result, is sinusoidal in time and differs in phase by a constant amount from blade to blade. Under this assumption, the steady-state response of a perfectly tuned system X_t can be easily obtained. If amplitude and phase of the jth mass of a perfectly tuned system are B_j and θ_j, respectively,

$$x_{t_j} = B_j \cos(\omega t + \theta_j); \quad j = 1,2,\ldots,N \tag{2}$$

where ω and N are excitation frequency and the number of degrees of freedom. Let the steady-state response of a mistuned system differ from that of a tuned system by δX, i.e.,

$$X = X_t + \delta X \tag{3}$$

Now, X_t satisfies the following differential equation:

$$M_t \ddot{X}_t + C\dot{X}_t + K_t X_t = F_r \tag{4}$$

where F_r is the external force vector corresponding to the rth engine order excitation for which the phase difference between adjacent blades' excitation is equal to $2\pi r$ divided by the number of blades. Similarly, for the mistuned system,

$$(M_t + \delta M)(\ddot{X}_t + \delta\ddot{X}) + C(\dot{X}_t + \delta\dot{X}) + (K_t + \delta K)(X_t + \delta X) = F_r \tag{5}$$

Subtracting Eq. (4) from Eq. (5),

$$(M_t + \delta M)\delta\ddot{X} + (K_t + \delta K)\delta X + C\delta\dot{X} = -\delta K X_t - \delta M \ddot{X}_t \tag{6}$$

Since it is known that the steady-state response X is harmonic, δX would also be harmonic. Let

$$x_j = A_j \cos(\omega t + \phi_j) \tag{7}$$

and

$$\delta x_j = \delta A_j \cos(\omega t + \epsilon_j) \tag{8}$$

Substituting expressions for δX and X_t into Eq. (6) and then equating the components of $\cos\omega t$ and $\sin\omega t$ on both sides, a system of $2N$ algebraic equations is obtained. These equations can be described as follows:

$$(H + \delta H) U = (Q + R) V \tag{9}$$

where

$$U = [\delta A_1 \cos\epsilon_1 \;\; \delta A_1 \sin\epsilon_1 \ldots \delta A_N \cos\epsilon_N \;\; \delta A_N \sin\epsilon_N]^T \tag{10}$$

$$V = [B_1 \cos\theta_1 \;\; B_1 \sin\theta_1 \ldots B_N \cos\theta_N \;\; B_N \sin\theta_N]^T \tag{11}$$

The matrix H consists of the elements of M_t, K_t, and C, whereas the matrix δH consists of the elements of δK and δM. The matrices Q and R contain the elements of δK and δM, respectively. Also, note that the matrix H that corresponds to a perfectly tuned system is circular in nature.

Since the matrix H is nonsingular for a damped system,

$$U = (I + H^{-1}\delta H)^{-1} H^{-1}(Q + R) V \tag{12}$$

Assuming that $H^{-1}\delta H$ is small,

$$(I + H^{-1}\delta H)^{-1} \approx I \tag{13}$$

Therefore, from Eq. (12),

$$U = H^{-1}(Q + R) V \tag{14}$$

The validity of the assumption (13) depends on the magnitude of the elements of $H^{-1}\delta H$, which is related to the excitation frequency, the damping factor, and the amount of mistuning. The worst situation corresponds to the resonant excitation frequency at which the elements of H^{-1} may become large if the damping is very low. In this paper, Eq. (13) is found to be valid (see Sec. III) under resonant conditions for the small amount of mistuning at damping levels typical of those observed in turbine rotor stages.

From Eq. (14)

$$u_p = \sum_{\ell=1}^{2N} \sum_{j=1}^{2N} a_{p\ell}(q_{\ell j} + r_{\ell j}) v_j, \quad p = 1,2,\ldots,2N \tag{15}$$

where $(H^{-1})_{p\ell} = a_{p\ell}$, $(Q)_{\ell j} = q_{\ell j}$, $(V)_j = v_j$, and $(U)_p = u_p$.

Since $E(q_{\ell j}) = E(r_{\ell j}) = 0$, from Eq. (15),

$$E(u_p) = 0, \quad p = 1,2,\ldots,2N \tag{16}$$

where $E(\;)$ is the expected value.

Each of the elements (u_p) of vector U is a linear combination of random variables $q_{\ell j}$ and $r_{\ell j}$ [Eq. (15)]. Since the distributions of these random variables are Gaussian and independent, u_p will have Gaussian distribution.[11] Furthermore, the random variable $z_j\,(= e u_{2j} + h u_{2j-1})$ can be shown to be Gaussian for all values of constants e and h. Therefore, the joint distribution of u_{2j-1} and u_{2j} will be Gaussian, i.e.,

$$f(u_{2j-1}, u_{2j}) = \frac{1}{2\pi\sigma_{2j-1}\sigma_{2j}(1-\gamma_j^2)^{1/2}} \exp\left[-\frac{1}{2(1-\gamma_j)^2} \times \left(\frac{u_{2j-1}^2}{\sigma_{2j-1}^2} - \frac{2\gamma_j u_{2j-1} u_{2j}}{\sigma_{2j-1}\sigma_{2j}} + \frac{u_{2j}^2}{\sigma_{2j}^2}\right)\right] \tag{17}$$

where σ_p is the standard deviation of u_p and γ_j the correlation coefficient of u_{2j-1} and u_{2j}, i.e.,

$$\sigma_p^2 = E(u_p^2) \tag{18}$$

and

$$\gamma_j = E(u_{2j-1} u_{2j}) / (\sigma_{2j-1}\sigma_{2j}) \tag{19}$$

The standard deviations σ_p and the correlation coefficients γ_j can be calculated analytically by developing appropriate expressions for them (see Appendix A).

The probability density functions of δA_j and A_j can be calculated using Eq. (17) as follows. From Eq. (10),

$$u_{2j-1} = \delta A_j \cos\epsilon_j \tag{20}$$

and

$$u_{2j} = \delta A_j \sin\epsilon_j \tag{21}$$

Therefore, the probability density function of δA_j is

$$g(\delta A_j) = \int_0^{2\pi} \delta A_j f(\delta A_j \cos\epsilon_j, \delta A_j \sin\epsilon_j) d\epsilon_j \tag{22}$$

Substituting Eq. (17) into Eq. (22),

$$g(\delta A_j) = \frac{\delta A_j}{\sigma_{2j-1}\sigma_{2j}(1-\gamma_j^2)^{1/2}} \times \exp\left[-\frac{(\delta A_j)^2}{4(1-\gamma_j^2)}\left(\frac{1}{\sigma_{2j-1}^2}+\frac{1}{\sigma_{2j}^2}\right)\right] I_0(q) \tag{23}$$

where I_0 is the modified Bessel function of the first kind and

$$q = \frac{(\delta A_j)^2}{2(1-\gamma_j^2)}\left[\frac{1}{4}\left(\frac{1}{\sigma_{2j-1}^2}-\frac{1}{\sigma_{2j}^2}\right)^2 + \left(\frac{\gamma_j}{\sigma_{2j-1}\cdot\sigma_{2j}}\right)^2\right]^{1/2} \tag{24}$$

To get the probability density function $g(A_j)$ of the amplitude (A_j), the following relationships are first obtained using Eqs. (3), (7), (8), (10), and (11),

$$A_j \cos\phi_j = v_{2j-1} + u_{2j-1} \tag{25}$$

and

$$A_j \sin\phi_j = v_{2j} + u_{2j} \tag{26}$$

Since u_{2j-1} and u_{2j} are jointly Gaussian, the joint distributions of $A_j\cos\phi_j$ and $A_j\sin\phi_j$ will also be Gaussian. Using Eqs. (17), (25), and (26),

$$g(A_j) = \frac{A_j}{2\pi\sigma_{2j-1}\sigma_{2j}(1-\gamma_j^2)^{1/2}} \int_0^{2\pi} \exp\left[-\frac{z(\phi_j)}{2(1-\gamma_j^2)}\right] d\phi_j \tag{27}$$

where

$$z(\phi_j) = \frac{(A_j\cos\phi_j - v_{2j-1})^2}{\sigma_{2j-1}^2} - \frac{2\gamma_j(A_j\cos\phi_j - v_{2j-1})(A_j\sin\phi_j - v_{2j})}{\sigma_{2j-1}\sigma_{2j}} + \frac{(A_j\sin\phi_j - v_{2j})^2}{\sigma_{2j}^2}$$

The probability P_r that A_j will be less than a critical value A_c can be calculated using Eq. (27) as follows:

$$Pr(A_j < A_c) = \left(\int_0^{Ac} g(A_j) dA_j\right) \Big/ \left(\int_0^{\infty} g(A_j) dA_j\right) \tag{28}$$

III. Comparison with Numerical Simulations

In order to verify the accuracy of this analytical technique, the results from numerical simulations are compared with those obtained analytically. A simple discrete model of a bladed disk assembly[12,13] is chosen for this purpose, see Fig. 1. The modal mass of each blade is considered to be identical and represented by m_t. The phenomenon of mistuning is simulated by considering the variation in modal stiffness alone. However, the standard deviation for the stiffness of each blade is chosen to be the same. In Fig. 1, k_j is the modal stiffness of the jth blade. The mechanical coupling between adjacent blades due to the disk's flexibility has been represented by a spring with stiffness kC. The nominal values of modal parameters are chosen from Ref. 12; see Table 1. The number of blades is selected to be equal to eight. And, the excitation frequency is chosen to represent the resonance condition; see Table 2.

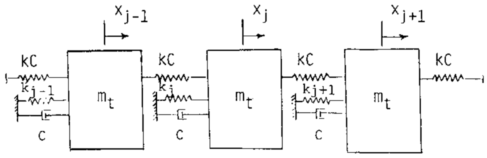

Fig. 1 Model of bladed disk assembly.

The IMSL subroutine[14] GGNML is used to choose the modal stiffness of each blade from a normal population with specified mean and standard deviation. The response of the resulting bladed disk assembly is obtained by solving the linear system of Eqs. (B3) and (B4). The statistics of the amplitude of each blade is then generated by considering a large number (arbitrarily chosen to be 1000) of bladed disk assemblies with different sets of blades' modal stiffness. The probability density function of δA_j is constructed by counting the number of occurrences in various intervals between zero and the maximum value of δA_j. The corresponding values for the same interval are obtained analytically using Eq. (23). First, standard deviations σ_p $(p=1,2,\ldots,2N)$, and correlation coefficients γ_p $(p=1,2,\ldots,N)$ are calculated using Eqs. (A6) and (A7). Then, the following expression is used to calculate the number of occurrences in various intervals,

$$NN_i = \left(\int_{z_i}^{z_{i+1}} g(z)dz\right) \Big/ \left(\int_0^{\infty} g(z)dz\right) \times NS \tag{29}$$

where NN_i is the number of occurrences in the interval $z_i < z < z_{i+1}$, NS the total number of simulations, and $z = \delta A_j$.

First, the standard deviation of the stiffness of each blade is chosen to be equal to 2000. The maximum value of δA_j is found to be about 40% of the amplitude of the corresponding tuned system response. The values of NN_i are calculated for all the blades using Eq. (29) and the numerical simulations as well. The results for one of the blades are presented in Table 3. Using the chi-square test of the hypothesis,[15] it has been found that the probability density functions of the

Table 1 System parameters (SI units)

$m_t = 0.0114$
$k_t = 430000.0$
$kC = 45430.0$

Table 2 Common values for all simulations

$NS = 1000$
$E(\delta k_j) = 0$
$\omega = 6328.8$

Table 3 Comparison with numerical simulations (SI units) $C = 1.381$ and $E(\delta K_j^2) = 4.0 \times 10^6$

Interval $\delta A \times 10^5$	No. of occurrences: Numerical simulations	No. of occurrences: Analytical results
0–0.35	76	74
0.35–0.70	176	186
0.70–1.05	221	223
1.05–1.40	190	194
1.40–1.75	160	139
1.75–2.10	80	87
2.10–2.45	49	50
2.45–2.80	27	26
2.80–3.15	15	13
3.15–3.50	5	6

Table 4 Comparison with numerical simulations (SI units) $C = 1.381$ and $E(\delta K_j^2) = 9.0 \times 10^6$

	No. of occurrences	
Interval $\delta A \times 10^5$	Numerical simulations	Analytical results
0.0–0.544	92	79
0.544–1.088	227	197
1.088–1.632	227	230
1.632–2.776	192	195
2.776–2.72	123	135
2.72–3.264	71	82
3.264–3.808	42	45
3.808–4.352	15	23
4.352–4.896	9	10
4.896–5.44	2	4
5.44–∞	0	1

Table 5 Comparison with numerical simulations (SI units) $C = 1.45$ and $E(\delta K_j^2) = 16 \times 10^6$

	No. of occurrences	
Interval $\delta A \times 10^5$	Numerical simulations	Analytical results
0.0–0.636	87	74
0.636–1.272	218	186
1.272–1.908	238	222
1.908–2.544	182	194
2.544–3.180	123	139
3.180–3.816	82	87
3.816–4.452	42	50
4.452–5.088	18	26
5.088–5.724	8	13
5.724–6.36	2	6
6.36–∞	0	3

Table 6 Comparison with numerical simulations (SI units) $C = 1.75$ and $E(\delta K_j^2) = 16 \times 10^6$

	No. of occurrences	
Interval $\delta A \times 10^5$	Numerical simulations	Analytical results
0.0–0.479	104	86
0.479–0.958	211	210
0.958–1.437	216	238
1.437–1.916	201	194
1.916–2.395	138	129
2.395–2.874	72	75
2.874–3.353	38	39
3.353–3.832	10	19
3.832–4.311	8	8
4.311–4.790	2	3
4.79–∞	0	1

amplitudes given by the analytical technique compare well with those from numerical simulations.

Next, a higher value (3000) of the standard deviation of the stiffness of each blade is considered. The results for one of the blades are presented in Table 4. The maximum value of δA_j is found to be about 56% of the amplitude of the tuned system response. The chi-square test of the hypothesis again indicates that the probability density functions of δA_j are accurate. However, the accuracy in this case is slightly lower as compared to that for the aforementioned case where the standard deviation of each blade's stiffness is equal to 2000. This can be attributed to the higher value of the elements of $H^{-1}\delta H$ [Eq. (13)].

Finally, the standard deviation is chosen to be equal to 4000 and a slightly higher value of damping ($C = 1.45$) is considered. This value of C corresponds to a 1% damping factor, which is typically observed in engines from aerodynamic and structural sources.[6,7] The comparison between results from numerical simulations and analytical technique are presented in Table 5. The chi-square test again verifies the accuracy of the analytical technique. However, the accuracy in this case is lower as compared to the results in Tables 3 and 4. This can be again attributed to the higher values of the elements of $H^{-1}\delta H$. Therefore, if the damping level is increased, the magnitude of the elements of H^{-1} would decrease and the analytical technique would be much more accurate. For example, an excellent agreement between numerical simulations and analytical technique can be seen in Table 6 for $C = 1.75$. It may be noted that this increase in the damping level has a practical significance because the damping factor in bladed disk assemblies is often greater than 1% because of additional damping sources, e.g., dry friction.[12,13]

IV. Conclusions

An approximate analytical technique to calculate the statistics of the forced response of a mistuned bladed disk assembly has been presented. This technique yields complete information about the probability density function of the amplitude of each blade. Therefore, the probability that the amplitude of a blade will exceed a certain critical value can be easily calculated. However, the accuracy of the method depends on the level of damping. The higher the damping, the greater the accuracy of the method. The comparisons with the results from numerical simulations of the responses of an eight-bladed mistuned bladed disk assembly indicates that the analytical technique is accurate for a small amount of mistuning at damping levels typical of those found in applications.

Appendix A: Calculation of $E(u_p^2)$ and $E(u_p u_s)$

From Eq. (15),

$$u_p = \sum_{j=1}^{4N^2} t_{pj} \tag{A1}$$

where t_{pj} is the jth term in Eq. (15). Therefore,

$$E(u_p^2) = \sum_{\ell=1}^{4N^2} \sum_{j=1}^{4N^2} E(t_{p\ell} t_{pj}) \tag{A2}$$

$$E(u_p u_s) = \sum_{\ell=1}^{4N^2} \sum_{j=1}^{4N^2} E(t_{p\ell} t_{sj}) \tag{A3}$$

Since $E(q_{\ell j}) = 0$, it is obvious that $E(t_{pj}) = 0$; see Eq. (15). Because of the fact that the distribution of a term in the right-hand side of Eq. (A1) is not dependent on all the remaining terms, many terms in Eqs. (A2) and (A3) would be equal to zero. And, the nonzero terms in Eqs. (A2) and (A3) can be expressed in terms of the standard deviation of the elements of matrices δK and δM. As an example, consider the model shown in Fig. 1. For this case,

$$(\delta K)_{ij} = 0 \quad \text{if} \quad i \neq j \tag{A4}$$

and $\delta M = 0$.

Therefore, from Eq. (15)

$$u_p = \sum_{j=1}^{N} (\delta K)_{jj} (a_{p,2j-1} v_{2j-1} + a_{p,2j} v_{2j}) \tag{A5}$$

And, it can be shown that

$$E(u_p^2) = \sum_{j=1}^{N} (a_{p,2j-1} v_{2j-1} + a_{p,2j} v_{2j})^2 E(\delta k_j^2) \tag{A6}$$

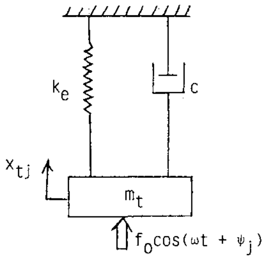

Fig. A1 Equivalent single degree-of-freedom model.

$$E(u_p u_s) = \sum_{j=1}^{N} (a_{p,2j-1} v_{2j-1} + a_{p,2j} v_{2j})$$

$$\times (a_{s,2j-1} v_{2j-1} + a_{s,2j} v_{2j}) E(\delta k_j^2) \qquad \text{(A7)}$$

Note that $a_{ij} = (H^{-1})_{ij}$. The matrix H for the system shown in Fig. 1 is

$$H = \begin{bmatrix} h1 & h3 & -h2 & 0 & 0 & 0 & \cdots & -h2 & 0 \\ -h3 & h1 & 0 & -h2 & 0 & 0 & \cdots & 0 & -h2 \\ -h2 & 0 & h1 & h3 & -h2 & 0 & \cdots & 0 & 0 \\ 0 & -h2 & -h3 & h1 & 0 & -h2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ -h2 & 0 & 0 & 0 & 0 & 0 & \cdots & h1 & h3 \\ 0 & -h2 & 0 & 0 & 0 & 0 & \cdots & -h3 & h1 \end{bmatrix}$$

where

$$h1 = k_t + 2kC - m_t\omega^2, \qquad h2 = kC, \qquad h3 = -c\omega$$

Also, the response of the tuned system v_j, which appears in Eqs. (A6) and (A7), can be calculated by analyzing the equivalent single-blade model.[13] See Fig. A1. For the system shown in Fig. 1

$$v_{2j-1} = \frac{(k_e - \omega^2 m_t) f_0 \cos\psi_j + c\omega f_0 \sin\psi_j}{(k_e - \omega^2 m_t)^2 + c^2\omega^2} \qquad \text{(A8)}$$

$$v_{2j} = \frac{-c\omega f_0 \cos\psi_j + (k_e - \omega^2 m_t) f_0 \sin\psi_j}{(k_e - \omega^2 m_t)^2 + c^2\omega^2} \qquad \text{(A9)}$$

where

$$k_e = k_t + 4kC \sin^2(\beta_r/2)$$

$$\beta_r = 2\pi r/NB, \quad r = 0,1,2,\ldots,NB-1$$

and

$$\psi_j = \beta_r \cdot (j-1), \quad j = 1,2,\ldots,NB$$

Appendix B: Steady-State Response of a Mistuned Bladed Disk Model

The system of differential equations governing the dynamics of the bladed disk model depicted in Fig. 1 is described as follows:

$$m_t\ddot{x}_j + k_j x_j + kC(x_j - x_{j+1}) + kC(x_j - x_{j-1}) + c\dot{x}_j$$

$$= f_0 \cos(\omega t + \psi_j), \quad j = 1,2,\ldots,N \qquad \text{(B1)}$$

Let the steady-state response be

$$x_j = A_j \cos(\omega t + \phi_j) \qquad \text{(B2)}$$

Substituting Eq. (B2) into Eq. (B1) and equating the coefficients of $\cos\omega t$ and $\sin\omega t$ on both sides,

$$(k_j + 2kC - \omega^2 m_t) A_j \cos\phi_j - c\omega A_j \sin\phi_j$$

$$- kCA_{j+1}\cos\phi_{j+1} - kCA_{j-1}\cos\phi_{j-1} = f_0 \cos\psi_j \qquad \text{(B3)}$$

and

$$c\omega A_j \cos\phi_j + (k_j + 2kC - \omega^2 m_t) A_j \sin\phi_j$$

$$- kCA_{j+1}\sin\phi_{j+1} - kCA_{j-1}\sin\phi_{j-1} = f_0 \sin\psi_j \qquad \text{(B4)}$$

The system of $2N$ linear equations (B3) and (B4) are solved to find N amplitudes (A_j) and N phase angles (ϕ_j).

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